A nice tutorial on the application of Muirhead's inequality can be found at:
https://kheavan.files.wordpress.com/2010/06/muirhead-69859.pdf

Also solved by Arkady Alt, San Jose, CA; Michael Brozinsky, Central Islip, NY; Ed Gray, Highland Beach, FL; Toshihiro Shimizu, Kawasaki Japan; Albert Stadler, Herrliberg, Switzerland, and the proposer.

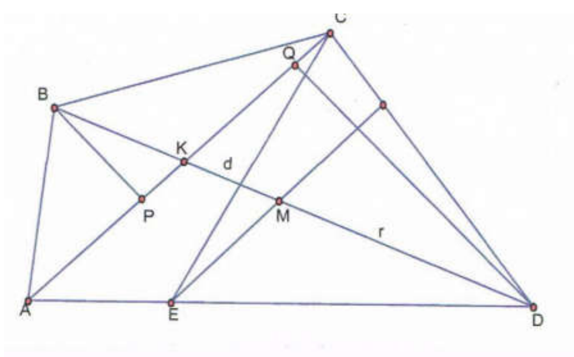
- **5393:** *Proposed by José Luis Díaz-Barrero, Barcelona, Tech, Barcelona, Spain*

Through the midpoint of the diagonal BD in the convex quadrilateral $ABCD$ we draw a straight line parallel to the diagonal AC. This line intersects the side AD at the point E. Show that

$$\frac{1}{[ABC]} + \frac{1}{[AEC]} \geq \frac{4}{[CED]}.$$

Here $[XYZ]$ represents the are of $\triangle XYZ$.

Solution 1 by Arkady Alt, San Jose, CA



We assume that midpoint M of diagonal BD does not coincide with K (the point of intersection of AC and BD) because otherwise $[AEC]=0$.

Also, w.l.o.g. assume that $KD>BK$.

Let $r=BM=MD$ and $d:=KM$. Since $ME \parallel AC$ then $\dfrac{AE}{ED}=\dfrac{KM}{MD}=\dfrac{d}{r}$ and, therefore,
$[AEC]=[ACD]\cdot\dfrac{AE}{AD}=[ACD]\cdot\dfrac{d}{r+d}$ and $[CED]=[ACD]\cdot\dfrac{ED}{AD}=[ACD]\cdot\dfrac{r}{r+d}$.

Let $BP, DQ\perp AC$.Then $\triangle KPB\approx\triangle KQD \implies \dfrac{BP}{DQ}=\dfrac{BK}{DK}$ and since
$BK=r-d, KD=r+d$ we have $\dfrac{[ABC]}{[ADC]}=\dfrac{AC\cdot BP}{AC\cdot DQ}=\dfrac{r-d}{r+d}$.

Thus, $\dfrac{1}{[ABC]}+\dfrac{1}{[AEC]}\geq\dfrac{4}{[CED]} \iff \dfrac{1}{[ABC]}+\dfrac{r+d}{d\cdot[ACD]}\geq\dfrac{4(r+d)}{r\cdot[ACD]} \iff$

$\dfrac{[ACD]}{[ABC]}+\dfrac{r+d}{d}\geq\dfrac{4(r+d)}{r} \iff \dfrac{r+d}{r-d}+\dfrac{r+d}{d}\geq\dfrac{4(r+d)}{r} \iff \dfrac{1}{r-d}+\dfrac{1}{d}\geq\dfrac{4}{r}$
and we have $\dfrac{1}{r-d}+\dfrac{1}{d}-\dfrac{4}{r}=\dfrac{(r-2d)^2}{dr(r-d)}\geq 0$.

Solution 2 by Toshihiro Shimizu, Kawasaki Japan

Let M be the midpoint of BD, F be the intersection of AC and BD and S be the area of the quadrilateral $ABCD$. Let $x=FB/DB$. Then, $[ABC]=xS$, $[ADC]=(1-x)S$ and $DM/DF=1/(2(1-x))$. Thus, $[AEC]=[ADC]\,|AE/AD|=[ADC]\,|MF/DF|=(1-x)S\,((1/2-x)/(1-x))=(1/2-x)\,S$. Similarly,
$[CED]=[ADC]\,|ED/AD|=[ADC]\,|MD/DF|=(1-x)S\,|(1/2)/(1-x)|=1/2S$.
Therefore we need to show that

$$\frac{1}{x}+\frac{1}{\frac{1}{2}-x}\geq\frac{4}{1/2}=8.$$

From Cauchy-Schwarz's inequality,

$$\left(\frac{1}{x}+\frac{1}{\frac{1}{2}-x}\right)\left(x+\frac{1}{2}-x\right)\geq 2^2$$

.It's equivalent to the desired inequality.

Soltuion 3 by Kee-Wai Lau, Hong Kong, China

Without loss of generality, let $\overline{BD}=2$. Let
$O=(0,0),\ A=(x_A,y_A),\ B=(0,1), C=(x_C,y_C)\ , D(0,-1)$, where $x_A>0$ and $x_C<0$.
Suppose that AC and BD interset at $F=(0,f)$. Since $OE||AC$ and E lies on AD, so $f>0$. Since quadrilateral $ABCD$ is convex, so $f<1$. Suppose that slope of $AC=$ slope of $OE=m$. We readily obtain $y_A=mx_A+f,\ y_C=mx_C+f$ and
$E=\left(\dfrac{x_A}{1+f},\dfrac{mx_A}{1+f}\right)$. By the standard formula, we obtain $[ABC]=\dfrac{(1-f)(x_A-x_C)}{2}$.
$[AEC]=\dfrac{f(x_A-x_C)}{2}$ and $[CED]=\dfrac{x_A-x_C}{2}$. Hence the inequality of the problem is